MATEMATICAS. BC2 TEMA 11: Métodos de integración

1. Resolver las siguientes integrales **por partes:**

1 $\int x \operatorname{sen} x \, dx$

2 $\int \frac{\ln x}{x^3} dx$

3 $\int (x^3 + 5x^2 - 2) e^{2x} dx$

4 $\int \ln x \, dx$

5 $\int e^x \cos x \, dx$

6 $\int x e^x \, dx$

7 $\int x^2 e^x \, dx$

8 $\int \operatorname{arc\,tg} x \, dx$

9 $\int e^x \operatorname{sen} x \, dx$

10 $\int x^2 \ln x \, dx$

11 $\int x^2 \operatorname{sen} 3x \, dx$

12 $\int \operatorname{arc\,sen} x \, dx$

2. Resolver las siguientes integrales **racionales:**

1 $\int \frac{3x^2 - 2x + 5}{(x+3)^3} dx$

2 $\int \frac{2x^2 + 5x - 1}{x^3 + x^2 - 2x} dx$

3 $\int \frac{x^2 + 1}{(x+1)^2 (x-3)} dx$

4 $\int \frac{x}{(x+1)(x^2 + x + 1)} dx$

5 $\int \frac{3x - 4}{x^2 - 2x + 4} dx$

3. Resolver las siguientes integrales **por cambio de variable:**

1 $\int x\sqrt{1+x} \, dx$

2 $\int \frac{dx}{\sqrt{1 + e^x}}$

3 $\int \frac{dx}{x\sqrt{x^2 - 2}}$

4 $\int \frac{\sqrt{x+1} + 2}{\sqrt[3]{(x+1)^2} - \sqrt{x+1}} dx$

5 $\int \operatorname{cosec}^3 x \, dx$

6 $\int \frac{e^{4x} + 3}{e^{3x}} dx$

7 $\int \frac{x^3 \sqrt{1+x^4}}{\sqrt{1+x^4} + 1} dx$

8 $\int \sqrt{1 - x^2} \, dx$

9 $\int \frac{dx}{\cos^4 x}$

10 $\int \frac{dx}{\cos^2 x \operatorname{sen}^2 x}$

11 $\int \frac{dx}{2 + \cos x}$

12 $\int \frac{1 + e^x}{1 - e^x} dx$

13 $\int \frac{3^x}{1 + 3^x} dx$

14 $\int \frac{dx}{\sqrt{4 - x^2}}$

15 $\int \sqrt{\frac{x+2}{x-1}} \, dx$

16 $\int \frac{dx}{1 + \operatorname{sen} x + \cos x}$

17 $\int \frac{4e^{3x}}{1 + e^{2x}} dx$

18 $\int \frac{dx}{\sqrt{\operatorname{sen} x \cos^3 x}}$

SOLUCIONES

Ejercicio 1:

1 $u = x \xrightarrow{\text{derivar}} u' = 1$

$v' = \operatorname{sen} x \xrightarrow{\text{integrar}} v = -\cos x$

$\int x \operatorname{sen} x \, dx = -x\cos x + \int \cos x \, dx = = -x\cos x + \operatorname{sen} x + C$

2 $u = \ln x \xrightarrow{\text{derivar}} u' = \frac{1}{x}$

$v' = \frac{1}{x^3} \xrightarrow{\text{integrar}} v = -\frac{1}{2x^2}$

$\int \frac{\ln x}{x^3} dx = -\frac{1}{2x^2}\ln x + \frac{1}{2}\int \frac{1}{x^3} dx = -\frac{1}{2x^2}\ln x - \frac{1}{4x^2} + C$

3 $u = x^3 + 5x^2 - 2 \xrightarrow{\text{derivar}} u' = 3x^2 + 10x$

$v' = e^{2x} \xrightarrow{\text{integrar}} v = \frac{1}{2}e^{2x}$

$\int (x^3 + 5x^2 - 2)\, e^{2x} dx = \frac{1}{2}(x^3 + 5x^2 - 2)e^{2x} - \frac{1}{2}\int (3x^2 + 10x)e^{2x} dx =$

$u = 3x^2 + 10x \xrightarrow{\text{derivar}} u' = 6x + 10$

$v' = e^{2x} \xrightarrow{\text{integrar}} v = \frac{1}{2}e^{2x}$

$= \frac{1}{2}(x^3 + 5x^2 - 2)e^{2x} - \frac{1}{2}\left(\frac{1}{2}(3x^2 + 10x)\, e^{2x} - \frac{1}{2}\int (6x + 10)e^{2x}\, dx\right) =$

$u = 6x + 10 \xrightarrow{\text{derivar}} u' = 6$

$v' = e^{2x} \xrightarrow{\text{integrar}} v = \frac{1}{2}e^{2x}$

$= \frac{1}{2}(x^3 + 5x^2 - 2)e^{2x} - \frac{1}{4}(3x^2 + 10x)e^{2x} + \frac{1}{8}(6x + 10)e^{2x} + \frac{3}{4}\int e^{2x} dx =$

$= \frac{1}{2}(x^3 + 5x^2 - 2)e^{2x} - \frac{1}{4}(3x^2 + 10x)e^{2x} + \frac{1}{8}(6x + 10)e^{2x} - \frac{3}{8}e^{2x} + C =$

$= \left(\frac{1}{2}x^3 + \frac{7}{4}x^2 - \frac{7}{4}x - \frac{1}{8}\right)e^{2x} + C$

4 $u = \ln x \xrightarrow{\text{derivar}} u' = \frac{1}{x}$

$v' = 1 \xrightarrow{\text{integrar}} v = x$

$\int \ln x \, dx = x\ln x - \int dx = = x\ln x - x + C$

5 $u = e^x \xrightarrow{\text{derivar}} u' = e^x$

$v' = \cos x \xrightarrow{\text{integrar}} v = \operatorname{sen} x$

$\int e^x \cos x \; dx = e^x \operatorname{sen} x - \int e^x \operatorname{sen} x \; dx$

$u = e^x \xrightarrow{\text{derivar}} u' = e^x$

$v' = \operatorname{sen} x \xrightarrow{\text{integrar}} v = -\cos x$

$$\int e^x \cos x \, dx = e^x \operatorname{sen} x + e^x \cos x - \int e^x \cos x \, dx$$

$$2\int e^x \cos x \, dx = e^x \operatorname{sen} x + e^x \cos x \underset{=}{} \int e^x \cos x \, dx = \frac{1}{2} e^x (\operatorname{sen} x + \cos x) + C$$

6 $u = x \xrightarrow{derivar} u' = 1$

$v' = e^x \xrightarrow{integrar} v = e^x$

$$\int x e^x \, dx = x e^x - \int e^x \, dx = x e^x - e^x + C = e^x (x-1) + C$$

7 $\int x^2 e^x \, dx$

$u = x^2 \xrightarrow{derivar} u' = 2x$

$v' = e^x \xrightarrow{integrar} v = e^x$

$$\int x^2 e^x \, dx = x^2 e^x - 2\int x e^x \, dx$$

$u = x \xrightarrow{derivar} u' = 1$

$v' = e^x \xrightarrow{integrar} v = e^x$

$$\int x^2 e^x \, dx = x^2 e^x - 2\left(x e^x - \int e^x dx\right) = = x^2 e^x - 2x e^x + 2e^x + C = e^x (x^2 - 2x + 2) + C$$

8 $\int arc\ tg\ x \, dx$

$u = arc\ tg\ x \xrightarrow{derivar} u' = \dfrac{1}{1+x^2}$

$v' = 1 \xrightarrow{integrar} v = x$

$$\int arc\ tg\ x \, dx = x\, arc\ tg\ x - \int \frac{x}{1+x^2} dx = = x\, arc\ tg\ x - \frac{1}{2} \ln(1+x^2) + C$$

9 $\int e^x \operatorname{sen} x dx$

$u = \operatorname{sen} x \xrightarrow{derivar} u' = \cos x$

$v' = e^x \xrightarrow{integrar} v = e^x$

$$\int e^x \operatorname{sen} x dx = e^x \operatorname{sen} x - \int e^x \cos x dx$$

$u = \cos x \xrightarrow{derivar} u' = -\operatorname{sen} x$

$v' = e^x \xrightarrow{integrar} v = e^x$

$$\int e^x \operatorname{sen} x dx = e^x \operatorname{sen} x - e^x \cos x - \int e^x \operatorname{sen} x dx \underset{=}{} 2\int e^x \operatorname{sen} x dx = e^x \operatorname{sen} x - e^x \cos x$$

$$\int e^x \operatorname{sen} x dx = \frac{e^x \operatorname{sen} x - e^x \cos x}{2} + C = = \frac{1}{2} e^x (\operatorname{sen} x - \cos x) + C$$

10 $\int x^2 \ln x \, dx$

$u = \ln x \xrightarrow{derivar} u' = \dfrac{1}{x}$

$v' = x^2 \xrightarrow{integrar} v = \dfrac{x^3}{3}$

$$\frac{1}{3}x^3\ln x-\frac{1}{3}\int x^2\,dx=\frac{1}{3}x^3\ln x-\frac{1}{9}x^3+C= =\frac{1}{3}x^3\left(\ln x-\frac{1}{3}\right)+C$$

11 $\int x^2 \operatorname{sen} 3x\,dx$

$u=x^2 \xrightarrow{\text{derivar}} u'=2x$

$v'=\operatorname{sen}3x \xrightarrow{\text{integrar}} v=-\frac{1}{3}\cos 3x$

$$\int x^2\operatorname{sen}3x\,dx=-\frac{1}{3}x^2\cos 3x+\frac{2}{3}\int x\cos 3x\,dx$$

$u=x \xrightarrow{\text{derivar}} u'=1$

$v'=\cos 3x \xrightarrow{\text{integrar}} v=\frac{1}{3}\operatorname{sen}3x$

$$\int x^2\operatorname{sen}3x\,dx=-\frac{1}{3}x^2\cos 3x+\frac{2}{3}\left(\frac{1}{3}x\operatorname{sen}3x-\frac{1}{3}\int\operatorname{sen}3x\,dx\right)= =-\frac{1}{3}x^2\cos 3x+\frac{2}{9}x\operatorname{sen}3x+\frac{2}{27}\cos 3x+C$$

12 $\int \operatorname{arc\,sen} x\,dx$

$u=\operatorname{arc\,sen}x \xrightarrow{\text{derivar}} u'=\frac{1}{\sqrt{1-x^2}}$

$v'=1 \xrightarrow{\text{integrar}} v=x$

$$\int\operatorname{arc\,sen}x\,dx=x\operatorname{arc\,sen}x+\int\frac{x}{\sqrt{1-x^2}}dx= =x\operatorname{arc\,sen}x+\sqrt{1-x^2}+C$$

Ejercicio 2:

1 $\int\frac{3x^2-2x+5}{(x+3)^3}dx$

$$\frac{3x^2-2x+5}{(x+3)^3}=\frac{A}{x+3}+\frac{B}{(x+3)^2}+\frac{C}{(x+3)^3}$$

$$3x^2-2x+5=A(x+3)^2+B(x+3)+C$$

Para calcular A, B y C, sustituimos x por −3:

$x=-3$ $38=C$

Derivamos y volvemos a sustituir por −3:

$6x-2=2A(x+3)+B$

$x=-3$ $-20=B$

Volvemos a derivar:

$6=2A$ $A=3$

$$\int\frac{3x^2-2x+5}{(x+3)^3}dx=\int\frac{3}{x+3}dx-\int\frac{20}{(x+3)^2}dx+\int\frac{38}{(x+3)^3}dx= =3\ln(x+3)+\frac{20}{x+3}-\frac{19}{(x+3)^2}+C$$

También podemos hallar los coeficientes realizando las operaciones e igualando coeficientes:

$$3x^2-2x+5=Ax^2+(6A+B)x+9A+3B+C\;;\quad\begin{cases} & 3=A\\ -2=18+B & B=-20\\ 5=27+C & C=38\end{cases}$$

2 $\int\frac{2x^2+5x-1}{x^3+x^2-2x}dx$

$$\frac{2x^2+5x-1}{x^3+x^2-2x}=\frac{A}{x}+\frac{B}{x-1}+\frac{C}{x+2}$$

Se efectúa la suma:

$$\frac{2x^2+5x-1}{x^3+x^2-2x}=\frac{A(x-1)(x+2)+Bx(x+2)+Cx(x-1)}{x(x-1)(x+2)}$$

Como las dos fracciones tienen el mismo denominador, los numeradores han de ser iguales: $2x^2+5x-1=A(x-1)(x+2)+Bx(x+2)+Cx(x-1)$

Calculamos los coeficientes de A, B y C dando a la x los valores que anulan al denominador.

$x=0 \qquad -1=A(-1)(2) \qquad A=\frac{1}{2}$

$x=1 \qquad 6=B(1)(3) \qquad B=2$

$x=-2 \qquad -3=C(-2)(-3) \qquad C=-\frac{1}{2}$

Se calculan las integrales de las fracciones simples:

$$\int\frac{2x^2+5x-1}{x^3+x^2-2x}dx=\frac{1}{2}\int\frac{dx}{x}+2\int\frac{dx}{x-1}-\frac{1}{2}\int\frac{dx}{x+2}==\frac{1}{2}\ln(x)+2\ln(x-1)-\frac{1}{2}\ln(x+2)+C$$

Otra forma de hallar los coeficientes es realizando las operaciones e igualando coeficientes. $2x^2+5x-1=(A+B+C)x^2+(A+2B-C)x-2A$

Igualamos coeficientes: $\begin{cases}2=A+B+C\\5=A+2B-C\\-1=-2A\end{cases} \qquad A=\frac{1}{2} \quad B=2 \quad C=-\frac{1}{2}$

3 $\int\frac{x^2+1}{(x+1)^2(x-3)}dx$

$$\frac{x^2+1}{(x+1)^2(x-3)}=\frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{C}{x-3};\quad \frac{x^2+1}{(x+1)^2(x-3)}=\frac{A(x+1)(x-3)+B(x-3)+C(x+1)^2}{(x+1)^2(x-3)}$$

$$x^2+1=A(x+1)(x-3)+B(x-3)+C(x+1)^2$$

Para calcular los valores de A, B y C, damos a x los valores que anulan al denominador y otro más.

$x=-1 \qquad 2=-4B \qquad B=-\frac{1}{2}$

$x=3 \qquad 10=16C \qquad C=\frac{5}{8}$

$x=0 \qquad 1=-3A-3B+C \qquad A=\frac{3}{8}$

$$\int\frac{x^2+1}{(x+1)^2(x-3)}dx=\frac{3}{8}\int\frac{dx}{x+1}-\frac{1}{2}\int\frac{dx}{(x+1)^2}+\frac{5}{8}\int\frac{dx}{x-3}==\frac{3}{8}\ln(x+1)-\frac{1}{2(x+1)}+\frac{5}{8}\ln(x-3)+C$$

4 $\int\frac{x}{(x+1)(x^2+x+1)}dx$

$$\frac{x}{(x+1)(x^2+x-1)}=\frac{A}{(x+1)}+\frac{Mx+N}{x^2+z+1};\quad \frac{x}{(x+1)(x^2+x+1)}=\frac{A(x^2+x+1)+(Mx+N)(x+1)}{(x+1)(x^2+x+1)}$$

$$x=(A+M)x^2+(A+M+N)x+A+N$$

Igualamos los coeficientes de los dos miembros. $\begin{cases}0=A+M\\1=A+M+N\\0=A+N\end{cases} \qquad A=-1 \quad M=1 \quad N=1$

$$\int\frac{x}{(x+1)(x^2+x+1)}dx=-\int\frac{dx}{x+1}+\int\frac{x+1}{x^2+x+1}dx=$$

La primera integral es de tipo logarítmico y la segunda la tenemos que descomponer en dos, que serán de **tipo logarítmico y tipo arcotangente**.

Multiplicamos por 2 en la segunda integral para ir preparándola.

$$-\int\frac{dx}{x+1}+\frac{1}{2}\int\frac{2x+2}{x^2+x+1}dx=$$

El 2 del numerador de segunda integral lo transformamos en 1 + 1.

$$-\int \frac{dx}{x+1}+\frac{1}{2}\int \frac{2x+1+1}{x^2+x+1}dx$$

Descomponemos la segunda integral en otras dos.

$$-\int \frac{dx}{x+1}+\frac{1}{2}\left(\int \frac{2x+1}{x^2+x+1}dx+\int \frac{1}{x^2+x+1}dx\right)=$$

$$-\int \frac{dx}{x+1}+\frac{1}{2}\int \frac{2x+1+1}{x^2+x+1}dx+\frac{1}{2}\int \frac{1}{x^2+x+1}dx$$

Las dos primeras integrales son de tipo logarítmico.

$$=-\ln(x+1)+\frac{1}{2}\ln\left(x^2+x+1\right)+\frac{1}{2}\int \frac{1}{x^2+x+1}dx$$

La integral que nos queda es de tipo arcotangente.

Vamos a transformar el denominador de modo que podamos aplicar la fórmula de la integral del arcotangente.

Transformamos el denominador en un binomio al cuadrado.

$$\int \frac{1}{x^2+x+1}dx=\int \frac{1}{\left(x^2+x+\frac{1}{4}\right)-\frac{1}{4}+1}dx=\int \frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}dx=$$

Multiplicar numerador y denominador por 4/3 para obtener 1 en el denominador.
Dentro del binomio al cuadrado multiplicaremos por la raíz cuadrada de 4/3.

$$-\int \frac{\frac{4}{3}}{\left[\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right]^2+1}dx-\int \frac{\frac{2}{\sqrt{3}}\cdot\frac{2}{\sqrt{3}}}{\left[\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right]^2+1}dx-=\frac{2}{\sqrt{3}}\int \frac{\frac{2}{\sqrt{3}}}{1+\left(\frac{2}{\sqrt{3}}\frac{2x+1}{2}\right)^2}dx=\frac{2}{\sqrt{3}}\int \frac{\frac{2}{\sqrt{3}}}{1-\left(\frac{2x+1}{\sqrt{3}}\right)^2}dx==\frac{2}{\sqrt{3}}\operatorname{arc\,tg}\frac{2x+1}{\sqrt{3}}+C$$

$$\int \frac{x}{(x+1)\left(x^2+x+1\right)}dx=-\ln(x+1)+\frac{1}{2}\ln\left(x^2+x+1\right)+\frac{1}{2}\cdot\frac{2}{\sqrt{3}}\operatorname{arc\,tg}\frac{2x+1}{\sqrt{3}}+C=$$

$$=-\ln(x+1)+\frac{1}{2}\ln\left(x^2+x+1\right)+\frac{1}{\sqrt{3}}\operatorname{arc\,tg}\frac{2x+1}{\sqrt{3}}+C$$

5 $\int \frac{3x-4}{x^2+2x+4}dx$

Sumamos y restamos 3 en el numerador, descomponemos en dos fracciones y en la primera sacamos factor común 3.

$$\int \frac{3x+3-3-4}{x^2+2x+4}dx=3\int \frac{x+1}{x^2+2x+4}dx-7\int \frac{1}{x^2+2x+4}dx=$$

Multiplicamos y dividimos en la primera fracción por 2.

$$-\frac{3}{2}\int \frac{2x+2}{x^2+2x+4}dx-7\int \frac{1}{x^2+2x+4}dx-$$

$$=\frac{3}{2}\ln\left(x^2+2x+4\right)-7\int \frac{1}{x^2+2x+4}dx=\int \frac{1}{x^2+2x+4}dx$$

Vamos a transformar el denominador de modo que podamos aplicar la fórmula de la integral del arcotangente.

Transformamos el denominador en un binomio al cuadrado.

$$x^2+2x+4=x^2+2x+1+3=(x+1)^2+3=3\left[\left(\frac{x+1}{\sqrt{3}}\right)^2+1\right];\quad \int \frac{1}{x^2+2x+4}dx=\frac{1}{3}\int \frac{dx}{\left(\frac{x+1}{\sqrt{3}}\right)^2+1}$$

$$\frac{x+1}{\sqrt{3}}=t \qquad \frac{1}{\sqrt{3}}dx=dt$$

Realizamos un **cambio de variable**.

$$\frac{1}{3}\int \frac{dx}{\left(\frac{x+1}{\sqrt{3}}\right)^2+1}=\frac{1}{3}\int \frac{\sqrt{3}}{t^2+1}dt=\frac{\sqrt{3}}{3}\operatorname{arc\,tg} t+C=\frac{\sqrt{3}}{3}\operatorname{arc\,tg}\left(\frac{x+1}{\sqrt{3}}\right)+C$$

$$\int \frac{3x-4}{x^2+2x+4}dx=\frac{3}{2}\ln\left(x^2+2x+4\right)-\frac{7\sqrt{3}}{3}\operatorname{arc\,tg}\left(\frac{x+1}{\sqrt{3}}\right)+C$$

Ejercicio 3:

1 $\int x\sqrt{1+x}\,dx \quad 1+x=t^2 \quad x=t^2-1 \quad dx=2t\,dt$

$$\int (t^2-1)\cdot t\cdot 2t\,dt=\int (2t^4-2t^2)\,dt=\frac{2}{5}t^5-\frac{2}{3}t^3+C$$

$$t=\sqrt{1+x} \qquad \frac{2}{5}\left(\sqrt{1+x}\right)^5-\frac{2}{3}\left(\sqrt{1+x}\right)^3+C==\frac{2}{5}(1+x)^2\sqrt{1+x}-\frac{2}{3}(1+x)\sqrt{1+x}+C$$

2 $\int \frac{dx}{\sqrt{1+e^x}} \quad 1+e^x=t^2 \quad e^x=t^2-1 \quad e^x\,dx=2t\,dt \quad dx=\frac{2t\,dt}{t^2-1}$

$$\int \frac{2t\,dt}{(t^2-1)t}=2\int \frac{dt}{t^2-1}$$

$$\frac{1}{t^2-1}=\frac{A}{t+1}+\frac{B}{t-1} \qquad 1=A(t-1)+B(t+1)$$

$t=-1 \quad 1=-2A \quad A=-\frac{1}{2}; \quad t=1 \quad 1=2B \quad B=\frac{1}{2}$

$$2\int \frac{dt}{t^2-1}=-\int \frac{dt}{t+1}+\int \frac{dt}{t-1}=-\ln(t+1)+\ln(t-1)+C=\ln\left(\frac{t-1}{t+1}\right)+C$$

$$t=\sqrt{1+e^x} \qquad \int \frac{dx}{\sqrt{1+e^x}}=\ln\left(\frac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}+1}\right)+C$$

3 $\int \frac{dx}{x\sqrt{x^2-2}} \quad x=\sqrt{2}\sec t \quad dx=\sqrt{2}\sec t\,\mathrm{tg}\,t\,dt$

$$\int \frac{\sqrt{2}\sec t\,\mathrm{tg}\,t}{\sqrt{2}\sec t\sqrt{2\sec^2 t-2}}dt=\int \frac{\mathrm{tg}\,t}{\sqrt{2(\sec^2 t-1)}}dt=\int \frac{\mathrm{tg}\,t}{\sqrt{2}\,\mathrm{tg}\,t}dt==\frac{1}{\sqrt{2}}\int dt=\frac{1}{\sqrt{2}}t+C$$

$x=\sqrt{2}\sec t \quad x=\frac{\sqrt{2}}{\cos t} \quad \cos t=\frac{\sqrt{2}}{x} \quad t=\arccos\left(\frac{\sqrt{2}}{x}\right); \quad \int \frac{dx}{x\sqrt{x^2-2}}=\frac{1}{\sqrt{2}}\arccos\left(\frac{\sqrt{2}}{x}\right)+C$

4 $\int \frac{\sqrt{x+1}+2}{\sqrt[3]{(x+1)^2}-\sqrt{x+1}}dx \quad x+1=t^6 \quad dx=6t^5\,dt$

$$\int \frac{t^3+2}{t^4-t^3}6t^5\,dt=6\int \frac{t^5+2t^2}{t-1}dt=6\int \left(t^4+t^3+t^2+3t+\frac{3}{t-1}\right)dt=$$

$$=\frac{6}{5}t^5+\frac{3}{2}t^4+2t^3+9t^2+18\ln(t-1)+C \; ; \quad x+1=t^6 \quad t=\sqrt[6]{x+1}$$

$$=\frac{6}{5}\sqrt[6]{(x+1)^5}+\frac{3}{2}\sqrt[6]{(x+1)^4}+2\sqrt[6]{(x+1)^3}+9\sqrt[6]{(x+1)^2}+18\ln\left(\sqrt[6]{(x+1)}-1\right)+C=$$

$$=\frac{6}{5}\sqrt[6]{(x+1)^5}+\frac{3}{2}\sqrt[3]{(x+1)^2}+2\sqrt{x+1}+9\sqrt[3]{x+1}+18\ln\left(\sqrt[6]{(x+1)}-1\right)+C$$

5 $\int \mathrm{cosec}^3 x\,dx=\int \frac{dx}{\mathrm{sen}^3 x} \quad \mathrm{tg}\,\frac{x}{2}=t \quad dx=\frac{2\,dt}{1+t^2}$

$$\int \frac{1}{\left(\frac{2t}{1+t^2}\right)^3}\frac{2dt}{1+t^2}=\int \frac{(1+t^2)^2}{4t^3}dx=\int \frac{1+2t^2+t^4}{4t^3}dx=$$

$$= \frac{1}{4}\int \frac{dt}{t^3} + \frac{1}{2}\int \frac{dt}{t} + \frac{1}{4}\int t\,dt = -\frac{1}{8t^2} + \frac{1}{2}\ln t + \frac{1}{8}t^2 + C = \; = -\frac{1}{8\operatorname{tg}^2\frac{x}{2}} + \frac{1}{2}\ln\left(\operatorname{tg}\frac{x}{2}\right) + \frac{1}{8}\operatorname{tg}^2\frac{x}{2} + C$$

6 $\int \frac{e^{4x}+3}{e^{3x}}dx$ $\quad e^x = t \quad e^x dx = dt \quad dx = \frac{dt}{t}$

$$\int \frac{e^{4x}+3}{e^{3x}}dx = \int \frac{t^4+3}{t^3\cdot t}dt = \int \frac{t^4+3}{t^4}dt = \int dt + 3\int \frac{dt}{t^4} = t - \frac{1}{t^3} + C = \; = e^x - \frac{1}{e^{3x}} + C$$

7 $\int \frac{x^3\sqrt{1+x^4}}{\sqrt{1+x^4}+1}dx$ $\quad 1+x^4 = t^2 \quad 4x^3\,dx = 2t\,dt \quad dx = \frac{t}{2x^3}dt$

$$\int \frac{x^3 t}{t+1}\cdot\frac{t}{2x^3}dt = \frac{1}{2}\int \frac{t^2}{t+1}dt =$$

t^2 $\quad |\underline{t+1}$

$\underline{-t^2-t}$ $\quad t-1$

$-t$

$t+1$

1

$$= \frac{1}{2}\int\left(t-1+\frac{1}{t+1}\right)dt = \frac{1}{4}t^2 - \frac{1}{2}t + \frac{1}{2}\ln(t+1) + C = \; = \frac{1}{4}\left(1+x^4\right) - \frac{1}{2}\sqrt{1+x^4} + \frac{1}{2}\ln\left(\sqrt{1+x^4}+1\right) + C$$

8 $\int \sqrt{1-x^2}\,dx$ $\quad x = \operatorname{sen} t \quad dx = \cos t\,dt$

$$\int \sqrt{1-x^2}\,dx = \int \sqrt{1-\operatorname{sen}^2 t}\cos t\,dt = \int \cos^2 t\,dt =$$

$$= \int\left(\sqrt{\frac{1+\cos 2t}{2}}\right)^2 dt = \frac{1}{2}\int (1+\cos 2t)dt = \; = \frac{1}{2}t + \frac{1}{4}\operatorname{sen} 2t + C$$

$t = \operatorname{arc\,sen} x$; $\operatorname{sen} 2t = 2\operatorname{sen} t\cdot\cos t = 2\operatorname{sen}(\operatorname{arc\,sen} x)\cos(\operatorname{arc\,sen} x) = 2x\cdot\sqrt{1-x^2}$

$\cos(\operatorname{arc\,sen} x) = \sqrt{1-[\operatorname{sen}(\operatorname{arc\,sen} x)]^2} = \sqrt{1-x^2}$ $\quad \int \sqrt{1-x^2}\,dx = \frac{1}{2}\operatorname{arc\,sen} x + \frac{1}{2}x\cdot\sqrt{1-x^2} + C$

9 $\int \frac{dx}{\cos^4 x}$ $\quad \operatorname{tg} x = t \quad dx = \frac{dt}{1+t^2}$

$$\int\left(\sqrt{1+t^2}\right)^4\frac{dt}{1+t^2} = \int (1+t^2)dt = t + \frac{1}{3}t^3 + C = \operatorname{tg} x + \frac{1}{3}\operatorname{tg}^3 x + C$$

10 $\int \frac{dx}{\cos^2 x\operatorname{sen}^2 x}$ $\quad \operatorname{tg} x = t \quad dx = \frac{dt}{1+t^2}$

$$\int \frac{dx}{\cos^2 x\operatorname{sen}^2 x} = \int \frac{\frac{dt}{1+t^2}}{\left(\frac{1}{\sqrt{1+t^2}}\right)^2\left(\frac{t}{\sqrt{1+t^2}}\right)^2} = \int \frac{\frac{dt}{1+t^2}}{\frac{1}{1+t^2}\;\frac{t^2}{1+t^2}} = \; = \int \frac{dt}{t^2} = -\frac{1}{t} - C = -\frac{1}{\operatorname{tg} x} = -\operatorname{cotg} x - C$$

11 $\int \frac{dx}{2+\cos x}$ $\quad \operatorname{tg}\frac{x}{2} = t \quad dx = \frac{2dt}{1+t^2}$

$$\int \frac{\frac{2dt}{1+t^2}}{2+\frac{1-t^2}{1+t^2}} = \int \frac{\frac{2dt}{1+t^2}}{\frac{2+2t^2+1-t^2}{1+t^2}} = \int \frac{\frac{2dt}{1+t^2}}{\frac{3+t^2}{1+t^2}} = 2\int \frac{dt}{3+t^2} =$$

$$-\frac{2}{3}\int\frac{dt}{1+\left(\frac{t}{\sqrt{3}}\right)^2}-\frac{2}{3}\sqrt{3}\int\frac{\frac{1}{\sqrt{3}}}{1+\left(\frac{t}{\sqrt{3}}\right)^2}dt-\frac{2}{3}\sqrt{3}\,\text{arc tg}\left(\frac{t}{\sqrt{3}}\right)+C=\frac{2}{3}\sqrt{3}\,\text{arc tg}\left(\frac{1}{\sqrt{3}}\,\text{tg}\,\frac{x}{2}\right)+C$$

12 $\int\frac{1+e^x}{1-e^x}dx$ $\quad e^x=t \quad e^x dx=dt \quad dx=\frac{dt}{t}$

$$\int\frac{1+t}{1-t}\frac{dt}{t}=\int\frac{-1-t}{t(t-1)}dt\ ;\quad \frac{-1-t}{t(t-1)}=\frac{A}{t}+\frac{B}{t-1}$$

$-1-t=A(t-1)+Bt \quad t=1 \quad -2=B$

$t=0 \quad -1=-A \quad A=1$

$$\int\frac{dt}{t}-2\int\frac{dt}{t-1}=\ln t-2\ln(t-1)+C==\ln e^x-2\ln(e^x-1)+C=x-\ln(e^x-1)^2+C$$

13 $\int\frac{3^x}{1+3^x}dx$ $\quad 3^x=t \quad 3^x\ln 3\,dx=dt \quad dx=\frac{dt}{t\cdot\ln 3}$

$$\int\frac{t}{1+t}\frac{dt}{t\cdot\ln 3}=\frac{1}{\ln 3}\int\frac{dt}{1+t}=\frac{1}{\ln 3}\ln(1+t)+C==\frac{1}{\ln 3}\ln(1+3^x)+C$$

14 $\int\frac{dx}{\sqrt{4-x^2}}$ $\quad x=2\,\text{sen}\,t \quad dx=2\cos t\,dt$

$$\int\frac{2\cos t\,dt}{\sqrt{4-4\text{sen}^2 t}}=\int\frac{2\cos t\,dt}{2\cos t}=\int dt=t+C \quad x=2\,\text{sen}\,t \quad t=\text{arc sen}\,\frac{x}{2}$$

$$\int\frac{dx}{\sqrt{4-x^2}}=\text{arc sen}\,\frac{x}{2}+C$$

15 $\int\sqrt{\frac{x+2}{x-1}}\,dx$

$\frac{x+2}{x-1}=t^2\ ;\quad x+2=t^2x-t^2 \quad t^2x-x=t^2+2 \quad x(t^2-1)=t^2+2\ ;\quad dx=\frac{-6t}{(t^2-1)^2}dt$

Integramos por partes.

$$\int\sqrt{\frac{x+2}{x-1}}\,dx=\int\sqrt{t^2}\,\frac{-6t}{(t^2-1)^2}dt=\int\frac{-6t^2}{(t^2-1)^2}dt$$

$u=t \xrightarrow{\text{derivar}} u'=1$

$v'=\frac{-6t}{(t^2-1)^2} \xrightarrow{\text{integrar}} v=\frac{3}{t^2-1}$

$$\int\frac{-6t^2}{(t^2-1)^2}dt=\frac{3t}{t^2-1}-3\int\frac{dt}{t^2-1}$$

Se realiza la **integral racional**.

$$\frac{1}{t^2-1}=\frac{A}{t-1}+\frac{B}{t+1} \qquad 1=A(t+1)+B(t-1)\ ;$$

$t=1 \quad 1=2A \quad A=\frac{1}{2}$

$$\int\frac{-6t^2}{(t^2-1)^2}dt=\frac{3t}{t^2-1}-\frac{3}{2}\left(\int\frac{dt}{t-1}-\int\frac{dt}{t+1}\right)$$

$$\int\frac{-6t^2}{(t^2-1)^2}dt=\frac{3t}{t^2-1}-\frac{3}{2}\ln(t-1)+\frac{3}{2}\ln(t+1)+C$$

$$\frac{x-2}{x-1}=t^2 \qquad t=\sqrt{\frac{x+2}{x-1}}$$

$$\int\sqrt{\frac{x+2}{x-1}}\,dx=\frac{3\sqrt{\frac{x+2}{x-1}}}{\frac{x+2}{x-1}-1}-\frac{3}{2}\left[\ln\left(\sqrt{\frac{x+2}{x-1}}-1\right)-\ln\left(\sqrt{\frac{x+2}{x-1}}+1\right)\right]+C=$$

Aplicamos las **propiedades de los logaritmos**.

$$=(x-1)\sqrt{\frac{x+2}{x-1}}-\frac{3}{2}\ln\left(\frac{\sqrt{\frac{x+2}{x-1}}-1}{\sqrt{\frac{x+2}{x-1}}+1}\right)+C$$

16 $\int\frac{dx}{1+\operatorname{sen}x+\cos x}$ $\qquad t=\operatorname{tg}\frac{x}{2}$ $\qquad dt=\frac{2dt}{1+t^2}$

$$\int\frac{\frac{2dt}{1+t^2}}{1+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}}=\int\frac{2}{1+t^2+2t+1-t^2}\,dt=\int\frac{2}{2+2t}\,dt=$$

$$=\int\frac{1}{1+t}\,dt=\ln(1+t)+C=\ln\left(1+\operatorname{tg}\frac{x}{2}\right)+C$$

17 $\int\frac{4e^{3x}}{1+e^{2x}}\,dx$ $\qquad e^x=t$ $\qquad e^x dx=dt$ $\qquad dx=\frac{dt}{t}$

$$\int\frac{4e^{3x}}{1+e^{2x}}\,dx=4\int\frac{t^3}{(1+t^2)\,t}\,dt=4\int\frac{t^2}{1+t^2}\,dt=4\int\frac{t^2+1-1}{1+t^2}\,dt=$$

$$4\left(\int dt-\int\frac{1}{1+t^2}\,dt\right)=4(t-\operatorname{arc\,tg}t)+C==4\left(e^x-\operatorname{arc\,tg}e^x\right)+C$$

18 $\int\frac{dx}{\sqrt{\operatorname{sen}x\cos^3x}}$ $\qquad \operatorname{tg}x=t$ $\qquad dx=\frac{dt}{1+t^2}$

$$\int\frac{\frac{dt}{1+t^2}}{\sqrt{\frac{t}{\sqrt{1+t^2}}\cdot\frac{1}{(1+t^2)\sqrt{1+t^2}}}}=\int\frac{\frac{dt}{1+t^2}}{\sqrt{\frac{t}{(1+t^2)^2}}}=\int\frac{\frac{dt}{1+t^2}}{\frac{\sqrt{t}}{1+t^2}}=$$

$$=\int\frac{dt}{\sqrt{t}}=2\int\frac{dt}{2\sqrt{t}}=2\sqrt{t}+C=2\sqrt{\operatorname{tg}x}+C$$